Envelope Theorem:

$$\frac{dM(x(a),a)}{da} = \frac{\partial M(x,a)}{\partial a}\Big|_{x=x^*(a)} \quad \text{where} \quad x^* = \arg\max_x M(x(a))$$

For example:

$$p = a - bQ \quad \text{where} \quad Q = \sum q_i, \quad mc = c_i$$

1. case – monopoly:

$$\pi = (p-c)q = (a-bq-c)q \;\Rightarrow\; \text{F.O.C: } \frac{d\pi}{dq} = a - c - 2bq = 0$$

Therefore $q^* = \frac{a-c}{2b}$, $p^* = \frac{a+c}{2}$, $\pi^* = \frac{(a-c)^2}{4b}$

i) explicit comparative analysis:

$$\begin{aligned}\frac{d\pi^*(q(c),c)}{dc} &= \frac{d}{dc}[\{p^*(c)-c\}q^*(c)] \\ &= \frac{d\{p^*(c)-c\}}{dc}q^*(c) + \{p^*(c)-c\}\frac{dq^*(c)}{dc} \\ &= (\frac{1}{2}-1)\frac{a-c}{2b} + \frac{a-c}{2}\frac{-1}{2b} \\ &= -\frac{(a-c)}{2b}\end{aligned}$$

ii) envelope theorem:

$$\frac{\partial\pi}{\partial c}\Big|_{q=q^*} = -q^* = -\frac{(a-c)}{2b}$$

From i) = ii), envelope theorem is valid in monopoly

2. case – duopoly:

$$\pi_i = (p-c_i)q_i = (a-b(q_i+q_j)-c_i)q_i \;\Rightarrow\; \text{F.O.C: } \frac{d\pi_i}{dq_i} = a - b(2q_i+q_j) - c_i = 0$$

$\Rightarrow$ Reaction function: $q_i = \frac{a-bq_j-c_i}{2b}$

Therefore, $q_i^* = \frac{a+c_j-2c_i}{3b}$, $p^* = \frac{a+c_i+c_j}{3}$, $\pi_i^* = \frac{(a+c_j-2c_i)^2}{9b}$

i) explicit comparative analysis:

$$\begin{aligned}\frac{d\pi^*(q_i(c_i,c_j),c_i)}{dc_i} &= \frac{d}{dc_i}[\{p^*(c_i,c_j)-c_i\}q_i^*(c_i,c_j)] \\ &= \frac{d\{p^*(c_i,c_j)-c_i\}}{dc_i}q_i^*(c_i,c_j) + \{p^*(c_i,c_j)-c_i\}\frac{dq_i^*(c_i,c_j)}{dc_i} \\ &= (\frac{1}{3}-1)\frac{a+c_j-2c_i}{3b} + \frac{a+c_j-2c_i}{3}\frac{-2}{3b} \\ &= -\frac{4(a+c_j-2c_i)}{9b}\end{aligned}$$

ii) envelope theorem:

$$\frac{d\pi_i}{dc_i}\Big|_{q_i=q_i^*,\,q_j=q_j^*} = -q_i^* = -\frac{a+c_j-2c_i}{3b}$$

From i) ≠ ii), envelope theorem is NOT valid in duopoly